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STABILITY OF LIQUID FLOW DOWN AN INCLINED TUBE

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Abstract—A linear stability analysis has been performed to investigate the stability of liquid flow down an inclined circular tube. To this purpose, approximate solutions which describe laminar and turbulent steady flow down an inclined tube have been developed first. The stability analysis has then been performed by an integral method. The results of the present investigation indicate that, in general, flow in a tube is more stable than in a channel and, in particular, there is a value of the liquid height at which the flow is always stable.

Key Words: linear stability, stratified flow, inclined tube

1. INTRODUCTION

The mechanism of formation of long wavelength disturbances at the surface of a liquid flowing down an inclined plane has been investigated extensively in the past (Kapitsa 1948; Yih 1954, 1963; Benjamin 1957; Hanratty & Hershman 1961).

The experimental observations have shown that the liquid interface is always agitated by waves, except for very low values of the film Re. Binnie (1957), for instance, observed that for water flowing down a vertical plane there are no evident waves for film Re <4.

Theoretical investigations have been carried out by means of a linear stability analysis of the Navier–Stokes equations describing film flow. A small disturbance

$$h' = h - \bar{h}, \qquad [1]$$

h and $\bar{h}$ being, respectively, the actual and the time-averaged value of the film height (see figure 1), of the form

$$h' = \hat{h}\exp[i\alpha(x - Ct)], \qquad [2]$$

is imposed on the interface in order to determine the conditions under which it becomes unstable. In [2], x is the coordinate along the main flow direction and t the time. The amplitude of the disturbance $\hat{h}$ and the wavenumber $\alpha = 2\pi/\lambda$ are real quantities, λ being the wavelength; the wave velocity $C = C_R + iC_I$ is complex. The wave will grow or decay in time depending on whether C_I is positive or negative. The condition $C_I = 0$, known as neutral stability, defines the transition from a stable to an unstable film. Substituting [1] and [2] into the Navier–Stokes equations, and linearizing, leads to the Orr–Sommerfeld equations. Various papers deal with their solution, using a variety of methods and approximations. Benjamin (1957), in particular, obtained by means of a power series approximation the stability condition relative to very long wavelength disturbances formed in liquid flow down an inclined plane.

An integral approach was outlined by Hanratty & Hershman (1961), who studied the stability of gas–liquid flow in a horizontal channel. According to this method, the Navier–Stokes equations are integrated in the direction normal to the free surface and the stability analysis is performed for the resulting averaged mass and momentum equations. Hanratty & Hershman (1961) showed that the method, when applied to study liquid flow down an inclined plane, gives a stability criterion very similar to that derived by Benjamin (1957).

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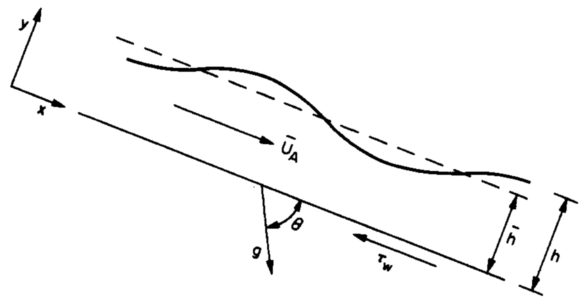

Figure 1. Model of the flow in the x-direction.

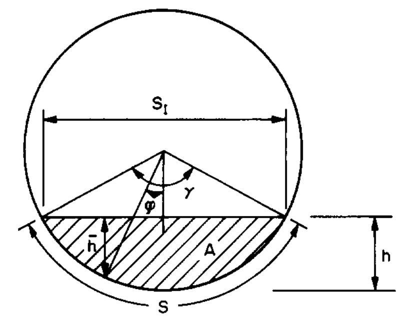

Figure 2. Model of the flow in the cross-section.

The parallel problem of flow down an inclined circular tube has not yet been considered in the literature. In the present paper, this problem will be solved adopting an integral approach similar to the one developed by Hanratty & Hershman (1961). By adopting this method, and an approximate solution to the problem of steady flow, both under laminar and turbulent conditions, it has been possible to derive the stability conditions in closed form.

2. PREVIOUS WORK

The stability condition derived by Benjamin (1957) can be stated as

$$\mathrm{Re} = 4\frac{\bar{U}_A \bar{h}}{\nu} < \tfrac{10}{3}\tan\Theta, \tag{3}$$

Θ being the inclination angle to the vertical, $\nu = \mu/\rho$ the kinematic viscosity, μ the viscosity and ρ the density of the liquid. The average velocity U_A is defined as

$$U_A = \frac{1}{h}\int_0^h u\,\mathrm{d}y, \tag{4}$$

where u is the velocity in the x-direction, y the coordinate normal to the free surface and $\bar{U}_A$ is the average velocity of the undisturbed flow.

Hanratty & Hershman (1961) characterized the velocity field by the shape factor Γ, defined as

$$\Gamma = \frac{1}{hU_A^2}\int_0^h u^2\,\mathrm{d}y. \tag{5}$$

They argued that a small disturbance introduced on the interface, produces disturbances in Γ, U_A and in the wall stress τ_W, that can be assumed linearly related to it, i.e.

$$\frac{h'}{\hat{h}} = \frac{U_A'}{\hat{U}_A} = \frac{\Gamma'}{\hat{\Gamma}} = \frac{\tau_W'}{\hat{\tau}_W} = \exp[i\alpha(x - Ct)], \tag{6}$$

where the amplitudes $\hat{U}_A$, $\hat{\tau}_W$ and $\hat{\Gamma}$ can be complex.

Assuming two-dimensional incompressible liquid flow and using a shallow liquid assumption, they considered the following integral mass and momentum balances:

$$\frac{\partial}{\partial x}\left(\int_0^h u\,\mathrm{d}y\right) + \frac{\partial h}{\partial t} = 0 \tag{7}$$

and

$$\frac{\partial}{\partial t}\left(\int_0^h u\,\mathrm{d}y\right) + \frac{\partial}{\partial x}\left(\int_0^h u^2\,\mathrm{d}y\right) = -\frac{\tau_W}{\rho} + gh\cos\Theta - gh\sin\Theta\frac{\partial h}{\partial x} + \frac{\sigma}{\rho}h\frac{\partial^3 h}{\partial x^3}, \tag{8}$$

where g is the acceleration due to gravity and σ the surface tension.

Substituting [4] and [5] into [7] and [8] and making use of [1] and [6], they found at neutral stability

$$\frac{\hat{U}_A}{\hat{h}} = \frac{C_R - \bar{U}_A}{\bar{h}}, \qquad [9]$$

$$\bar{U}_A^2\left[\left(\frac{C_R}{\bar{U}_A}\right)^2 - 2\bar{\Gamma}\frac{C_R}{\bar{U}_A} + \bar{\Gamma} - \frac{\bar{h}}{\hat{h}}\hat{\Gamma}_R\right] = g\bar{h}\sin\Theta + \frac{\hat{\tau}_{WI}}{\rho\alpha\hat{h}} + \frac{\sigma\alpha^2\bar{h}}{\rho} \qquad [10]$$

and

$$\frac{\hat{\tau}_{WR}}{\hat{h}} = \rho g\cos\Theta + \rho\alpha\bar{h}\,\bar{U}_A^2\frac{\hat{\Gamma}_I}{\hat{h}}. \qquad [11]$$

In order to solve this problem, Hanratty & Hershman (1961) used a pseudosteady-state approximation, whereby the instantaneous wall shear stress and the shape factor are related to flow variables by the same equations as derived for the undisturbed flow.

By solving the Navier–Stokes equations for steady laminar motion of a liquid down an inclined plane, we obtain

$$\tau_W = \frac{3\mu U_A}{h} \qquad [12]$$

and

$$\Gamma = \tfrac{6}{5}. \qquad [13]$$

From [9] and [11], it is finally obtained that

$$\frac{C_R}{\bar{U}_A} = 3 \qquad [14]$$

and, at neutral stability,

$$\mathrm{Fr} = \frac{1}{\sqrt{3}}\sqrt{\sin\Theta + \frac{\sigma\alpha^2}{\rho g}}, \qquad [15]$$

where Fr, defined as

$$\mathrm{Fr} = \frac{\bar{U}_A}{\sqrt{g\bar{h}}}, \qquad [16]$$

is the Froude number.

Steady flow can be expressed in terms of Fr as

$$\mathrm{Fr}^2 = \tfrac{1}{12}\,\mathrm{Re}\cos\Theta. \qquad [17]$$

Equation [17] combined with [15] gives

$$\mathrm{Re} = 4\left(\tan\Theta + \frac{\sigma\alpha^2}{\rho g\cos\Theta}\right). \qquad [18]$$

It is noteworthy that, except for the numerical factor, the stability criterion, for $\alpha \to 0$, is identical with the condition [3] obtained by solving the Orr–Sommerfeld's equations, therefore validating the approximate analysis based on the integral equations.

It can also be remarked that in Benjamin's paper, before approaching the general solution for small $(\alpha\bar{h})$, the following equations were derived:

$$\frac{C_R}{\bar{U}_A} = 3[1 - (\alpha\bar{h})^2 + \tfrac{11}{6}(\alpha\bar{h})^4 - 3.3555556\,(\alpha\bar{h})^6 + 0.0077581\,(\alpha\bar{h}\,\mathrm{Re})^2] \qquad [19]$$

and

$$\frac{1}{\mathrm{Re}}\left(\tan\Theta + \frac{\sigma\alpha^2}{\rho g\cos\Theta}\right) = [\tfrac{6}{5} - 4.1466856\,(\alpha\bar{h})^2 + 10.976471\,(\alpha\bar{h})^4 + 0.000047925\,(\alpha\bar{h}\,\mathrm{Re})^2]. \qquad [20]$$

The error in [19] is known to be considerably less than $O[(\alpha\bar{h}\ \mathrm{Re})^4]$. Equation [20] expresses the approximate relation between Re and α for neutral stability. The equation is easily solved, since for a specified value of Re, it becomes quadratic in $(\alpha\bar{h})^2$.

A simple analysis of the second member of [20] shows that, for small $(\alpha\bar{h})$ and Re, the quantity in square brackets is $<(6/5)$, i.e. closer to the value found with the integral method, further validating that analysis.

Also, Kapitsa (1948) used an integral approach, different from that of Hanratty–Hershman, and presented a second-order solution to the problem of flow down a vertical plane.

Adapting his approach to an inclined channel, the following neutral stability condition is obtained:

$$\mathrm{Fr} = \sqrt{\frac{5}{21}\left(\sin\Theta + \frac{\sigma\alpha^2}{\rho g}\right)}; \qquad [21]$$

or also

$$\mathrm{Re} = \frac{20}{7}\left(\tan\Theta + \frac{\sigma\alpha^2}{\rho g\cos\Theta}\right); \qquad [22]$$

still similar to the previous one, but less precise, as discussed by Yih (1963).

Yih solved the Orr–Sommerfeld equation by expansion in powers of $(\alpha\bar{h}\ \mathrm{Re})$. Initial erroneus numerical computations (Yih 1954) were corrected by solving the equation with a perturbation procedure, which gave results in complete agreement with those of Benjamin for long waves.

In principle, Benjamin's method is more accurate, but it cannot be used to solve other problems, in the sense that it can be applied only to very simple flow conditions. Yih's method is accurate and more adaptable, for example, to the cases of small waves or large Re, but Hanratty–Hershman's integral approach can be used, in addition, for turbulent flow (see Hanratty 1983) and for more complicated geometries than a channel (Lin & Hanratty 1986).

For a circular duct the average velocity and the shape factor can be defined as

$$U_{\mathrm{P}} = \frac{1}{A}\int_A u\,\mathrm{d}A \qquad [23]$$

and

$$\Gamma_{\mathrm{P}} = \frac{1}{AU_{\mathrm{P}}^2}\int_A u^2\,\mathrm{d}A, \qquad [24]$$

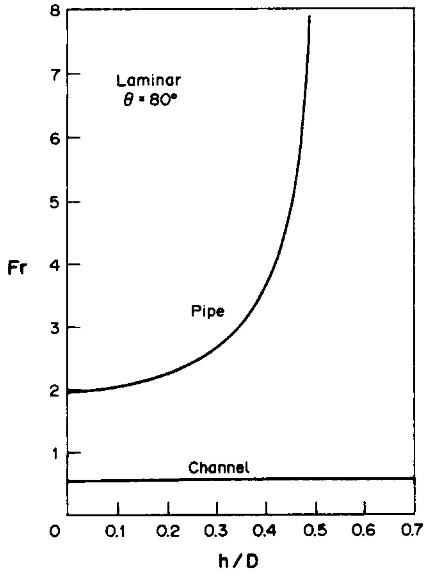

Figure 3. Flow stability in pipe and channel flow; laminar case.

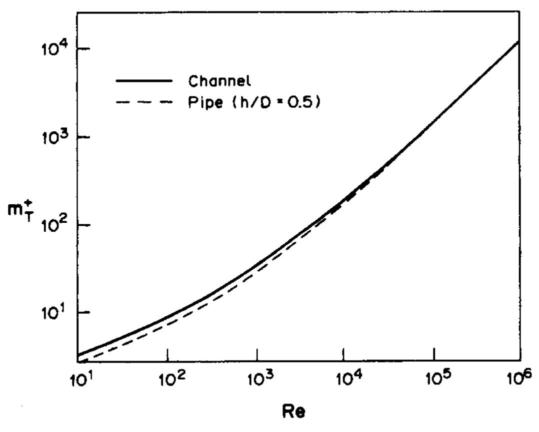

Figure 4. Analysis of steady, turbulent flow of a liquid layer in a pipe and in a channel.

where A is the cross-sectional area occupied by the liquid (see figure 2). The integral mass and momentum balances can now be written as (Lin & Hanratty 1986)

$$\frac{\partial(AU_P)}{\partial x} + \frac{\partial A}{\partial t} = 0 \qquad [25]$$

and

$$\frac{\partial(AU_P)}{\partial t} + \frac{\partial(A\Gamma_P U_P^2)}{\partial x} = -\frac{\tau_W S}{\rho} + gA\cos\Theta - gA\sin\Theta\frac{\partial h}{\partial x} + \frac{\sigma A}{\rho}\frac{\partial^3 h}{\partial x^3}, \qquad [26]$$

where S is the wetted perimeter.

At neutral stability it is found that

$$\frac{\hat{U}_P}{\hat{A}} = \frac{C_R - \bar{U}_P}{\bar{A}}, \qquad [27]$$

$$\bar{U}_P^2\left[\left(\frac{C_R}{\bar{U}_P}\right)^2 - 2\bar{\Gamma}_P\frac{C_R}{\bar{U}_P} + \bar{\Gamma}_P - \frac{\hat{\Gamma}_{PR}\bar{A}}{\hat{A}}\right] = g\frac{\bar{A}}{\bar{S}_I}\sin\Theta + \frac{\hat{\tau}_{WI}\bar{S}}{\rho\alpha\hat{A}} + \frac{\sigma\alpha^2\bar{A}}{\rho\bar{S}_I} \qquad [28]$$

and

$$\frac{\bar{\tau}_W\hat{S} + \hat{\tau}_{WR}\bar{S}}{\hat{A}} = \rho g\cos\Theta + \rho\alpha\bar{A}\,\bar{U}_P^2\frac{\hat{\Gamma}_{PI}}{\hat{A}}, \qquad [29]$$

S_I being the length of the free surface.

In order to adopt this method, a relation like [12] between the wall stress and the mean velocity for steady flow is needed. In the following sections, the aforementioned approach will be used to investigate the stability of flow down an inclined tube, both under laminar and turbulent conditions.

3. ANALYSIS

3.1. *Laminar flow*

Steady-state, free-falling laminar flow of an incompressible liquid layer down an inclined duct can be described by the Navier–Stokes equation

$$\mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = -\rho g\cos\Theta, \qquad [30]$$

with

$$u = 0, \quad \text{at the wall}, \qquad [31]$$

and

$$\frac{\partial u}{\partial y} = 0, \quad \text{at the free surface}. \qquad [32]$$

Analytical solutions of [30]–[32] can be obtained in a number of cases. In particular, the velocity profile relative to flow down an inclined plane or to annular flow in a vertical pipe (for $h \ll$ pipe radius) is given by

$$u = \frac{\tau_W}{\mu}y - \frac{1}{2}\frac{\rho}{\mu}gy^2\cos\Theta \qquad [33]$$

or, in terms of the dimensionless variables

$$u^+ = u\sqrt{\frac{\rho}{\tau_W}}, \quad y^+ = \frac{y}{\nu}\sqrt{\frac{\tau_W}{\rho}}, \quad h^+ = \frac{h}{\nu}\sqrt{\frac{\tau_W}{\rho}}, \qquad [34]$$

is given by

$$u^+ = y^+\left(1 - \frac{y^+}{2h^+}\right). \qquad [35]$$

The Re, defined by [3], can be expressed in terms of the dimensionless film height as

$$\mathrm{Re} = \frac{4}{3} h^{+2}. \qquad [36]$$

In a duct of arbitrary shape, the characteristic, or mean, thickness of the liquid layer can be defined as the ratio between the flow area occupied by the liquid and the wetted perimeter,

$$m = \frac{A}{S}. \qquad [37]$$

As the relations derived for flow in a channel are based on the assumption that the width of the channel is much larger than the film thickness, for a rectangular channel it can be assumed that

$$m \approx h. \qquad [38]$$

.The generalized definitions of the liquid Re and the dimensionless thickness, m^+, can then be introduced:

$$\mathrm{Re} = \frac{4AU_\mathrm{P}}{S\nu} \qquad [39]$$

and

$$m^+ = \frac{A}{S} \frac{1}{\nu} \sqrt{\frac{\tau_\mathrm{W}}{\rho}}. \qquad [40]$$

For a duct of arbitrary shape, it can be expected that the liquid Re is a function of m^+ and some dimensionless parameter describing the shape of the liquid phase cross-section.

In a duct of circular cross-section with diameter, exact solutions of [30]–[32] can be found for the following cases:

(*i*) *h/D = 0.5*

The axial velocity is, in this case, a function of the radial coordinate r only, then integration of [30] leads to

$$u = \frac{1}{4} \frac{\rho g \cos \Theta}{\mu} (h^2 - r^2). \qquad [41]$$

The wall stress τ_W can be calculated as

$$\tau_\mathrm{W} = \frac{h}{2} \rho g \cos \Theta \qquad [42]$$

and

$$U_\mathrm{P} = \frac{h}{4} \frac{\tau_\mathrm{W}}{\mu} = \frac{g \cos \Theta}{2\nu} \frac{A^2}{S^2}. \qquad [43]$$

The liquid Re can be obtained easily from the velocity profile and expressed as a function of m^+:

$$\mathrm{Re} = 2m^{+2}. \qquad [44]$$

(*ii*) *h/D* $\rightarrow 0$

For small values of h/D, it can be assumed [see, for instance, Russel *et al.* (1974) and Batchelor (1967)] that

$$\frac{\partial^2 u}{\partial z^2} \ll \frac{\partial^2 u}{\partial y^2}. \qquad [45]$$

Equation [30] can be integrated twice to give

$$u = \frac{g \cos \Theta}{2\nu} (2\tilde{h} y - y^2), \qquad [46]$$

where $\tilde{h}$ is given by

$$\tilde{h}(x, \varphi) = R\left(\cos\varphi - \cos\frac{\gamma}{2}\right) \quad [47]$$

and R is the tube radius, γ the angle which subtends the liquid film and the angle φ is defined in figure 2.

Then the volumetric flow rate is

$$Q = AU_{\mathrm{P}} = \int_0^{S_{\mathrm{I}}}\int_0^{\tilde{h}} u \,\mathrm{d}y\,\mathrm{d}z = \frac{g\cos\Theta R^4}{12\nu}\left[\frac{15}{2}(\gamma - \sin\gamma) - \sin^2\frac{\gamma}{2}(6\gamma - \sin\gamma)\right]. \quad [48]$$

Re in this case is obtained by means of [39] and [48]:

$$\mathrm{Re} = \frac{g\cos\Theta R^3}{3\nu^2}\left(\frac{\gamma - \sin\gamma}{\gamma}\right)\left[\frac{15}{2} - \sin^2\frac{\gamma}{2}\left(\frac{6\gamma - \sin\gamma}{\gamma - \sin\gamma}\right)\right]. \quad [49]$$

Substituting the wall stress

$$\tau_{\mathrm{W}} = \rho g\cos\Theta\frac{A}{S} = \rho g\cos\Theta\frac{R}{2}\left(\frac{\gamma - \sin\gamma}{\gamma}\right) \quad [50]$$

into [40] and making use of [49], Re assumes the following expression:

$$\mathrm{Re} = \frac{8}{3}m^{+2}\left(\frac{\gamma}{\gamma - \sin\gamma}\right)^2\left[\frac{15}{2} - \sin^2\frac{\gamma}{2}\left(\frac{6\gamma - \sin\gamma}{\gamma - \sin\gamma}\right)\right]. \quad [51]$$

In pipe flow, the solutions of both cases, $h/D = 0.5$ and $h/D \to 0$, can be expressed in the form

$$\mathrm{Re} = \tfrac{4}{3}\mathrm{a}_0 m^{+2}, \quad [52]$$

where a_0 is a numerical constant.

This equation is similar to the analytical solution obtained for stratified flow in a rectangular channel except that the coefficient a_0 in [52] depends on h/D or γ.

Fortunately, this coefficient is a *very weak* monotone function of h/D, at least in the range $0 < h/d < 0.5$, which is the most significant in practical applications.

A comparison between [52] and [44] gives, for $h/D = 0.5$,

$$\mathrm{a}_0 = \tfrac{3}{2} \quad [53]$$

and from [51], for $h/D \to 0$,

$$\mathrm{a}_0 = \lim_{\gamma\to 0^+} 2\left(\frac{\gamma}{\gamma - \sin\gamma}\right)^2\left[\frac{15}{2} - \sin^2\frac{\gamma}{2}\left(\frac{6\gamma - \sin\gamma}{\gamma - \sin\gamma}\right)\right] = \frac{54}{35}. \quad [54]$$

The ratio between the two values of a_0 is 105/108. In the full range $0 < h/D < 0.5$, [52] with $\mathrm{a}_0 = 3/2$, i.e.

$$\mathrm{Re} = 2m^{+2}, \quad [55]$$

gives an almost perfect fit to the numerical integration of the Navier–Stokes equations presented by Buffham (1968) in the form of a plot of the total flowrate vs the liquid height.

It has to be noted that the method proposed by Taitel & Dukler (1976) for the prediction of the hold-up in pipe flow, based on the equation

$$\tau_{\mathrm{W}} = \tfrac{1}{2}f\rho U_{\mathrm{P}}^2 \quad [56]$$

with the friction factor f given, for laminar flow, by

$$f = \frac{16}{\mathrm{Re}}, \quad [57]$$

is completely equivalent to using [55].

Equation [52], with [39] and [40], can be rewritten as

$$\tau_W = \frac{3\mu}{a_0}\frac{U_P S}{A}, \qquad [58]$$

hence, using [6] and [27] and assuming $a_0 = 3/2$, neutral stability conditions are expressed by

$$\frac{\hat{\tau}_{WR}}{\bar{\tau}_W} = \frac{C_R}{\bar{U}_P}\frac{\hat{A}}{\bar{A}} + \frac{\hat{S}}{\bar{S}} - 2\frac{\hat{A}}{\bar{A}} \qquad [59]$$

and

$$\hat{\tau}_{WI} = 0. \qquad [60]$$

The shape factor Γ_P can be easily calculated in the two limiting cases, from [41] and [46]:

(*i*) *for h/D = 0.5,*

$$\Gamma_P = 4/3; \qquad [61]$$

and

(*ii*) *for h/D → 0*

$$\Gamma_P = \left(\frac{g\cos\Theta}{\nu}\right)^2 \frac{A}{Q^2}\frac{R^6}{360}\left[\gamma\left(15 + 180\cos^2\frac{\gamma}{2} + 120\cos^4\frac{\gamma}{2}\right) - \sin\gamma\left(113 + 194\cos^2\frac{\gamma}{2} + 8\cos^4\frac{\gamma}{2}\right)\right]. \qquad [62]$$

From [62], the following limiting value can be obtained for $h/D \to 0$:

$$\lim_{\gamma\to 0^+} \Gamma_P = \frac{140}{99} = 1.\overline{41}. \qquad [63]$$

Using the relation

$$\gamma = 2\cos^{-1}\left(1 - 2\frac{h}{D}\right), \qquad [64]$$

it is also possible to calculate explicitly the derivative of the shape factor Γ_P with respect to the dimensionless height h/D of the liquid layer in the limit $h/D \to 0$:

$$\lim_{\gamma\to 0^+} \frac{\mathrm{d}\Gamma_P}{\mathrm{d}\left(\frac{h}{D}\right)} = -\frac{448}{3861} \approx -0.116. \qquad [65]$$

Interpolating Γ_P using the values given by [61] and [63] and its derivative [65], we obtain

$$\Gamma_P\left(\frac{h}{D}\right) = \frac{140}{99} - \frac{448}{3861}\frac{h}{D} - \frac{32}{351}\left(\frac{h}{D}\right)^2. \qquad [66]$$

With the help of the relation

$$\frac{\hat{A}}{\hat{h}} = \bar{S}_1, \qquad [67]$$

at neutral stability it is found that

$$\hat{\Gamma}_{PR} = -\left(\frac{448}{3861} + \frac{64}{351}\frac{\bar{h}}{D}\right)\frac{\hat{A}}{D\bar{S}_1} \qquad [68]$$

and

$$\hat{\Gamma}_{PI} = 0. \qquad [69]$$

Substituting [50], [59] and [69] in [29], we obtain the following equation for the nondimensional wave velocity:

$$\frac{C_R}{\bar{U}_P} = 3 - 2\frac{\hat{S}\bar{A}}{\bar{S}\hat{A}}. \qquad [70]$$

The Fr for pipe flow can be defined as

$$\mathrm{Fr} = \bar{U}_P\sqrt{\frac{\bar{S}}{g\bar{A}}}. \qquad [71]$$

Substituting [60] into [28], the Fr at neutral stability can be calculated as

$$\mathrm{Fr} = \sqrt{\frac{\frac{\bar{S}}{\bar{S}_I}\left(\sin\Theta + \frac{\sigma\alpha^2}{\rho g}\right)}{\left[\left(\frac{C_R}{\bar{U}_P}\right)^2 + \bar{\Gamma}_P - 2\bar{\Gamma}_P\left(\frac{C_R}{\bar{U}_P}\right) - \bar{A}\frac{\hat{\Gamma}_{PR}}{\hat{A}}\right]}}. \qquad [72]$$

The main difference with respect to channel flow is that the Fr is now a function of the liquid level h/D and, in particular, there is an asymptote over which the flow is always stable. The position of the asymptote depends mainly on the value of $\hat{\Gamma}_{PR}$ and on surface tension. In particular, it occurs at $h/D = 0.5$ if both $\hat{\Gamma}_{PR}$ and σ are neglected. Figure 3 shows the result obtained with $\hat{\Gamma}_{PR}$ computed from [68], σ being neglected.

In the limit $\gamma \to 0^+$, it is possible to calculate exactly the Fr at the transition. From [67], [70] and the relation

$$\hat{S} = \frac{2D}{\bar{S}_I}\hat{h}, \qquad [73]$$

we obtain

$$\frac{C_R}{\bar{U}_P} = 3 - 4\frac{\bar{A}D}{\bar{S}\bar{S}_I^2} = 3 - 2\frac{\gamma - \sin\gamma}{\gamma(1-\cos\gamma)}. \qquad [74]$$

It follows, from [72], that

$$\lim_{\gamma\to 0^+}\mathrm{Fr} = \sqrt{\frac{27}{7}\left(\sin\Theta + \frac{\sigma\alpha^2}{\rho g}\right)}. \qquad [75]$$

If surface tension effects are neglected, the Fr at neutral stability can be calculated from [15] and [75] for channel and tube flow, respectively. To give an example, at an inclination $\Theta = 80°$ we obtain

$$\mathrm{Fr} \approx 0.573 \quad \text{for the channel}$$

and

$$\mathrm{Fr} \approx 1.949 \quad \text{for the tube}.$$

3.2. Turbulent flow

Turbulent flow of a liquid layer has usually been described by means of eddy viscosity relations developed for full pipe flow. For instance, the Van Driest relation

$$\frac{\epsilon}{\nu} = 0.16\, y^{+2}\left[1 - \exp\left(-\frac{y^+}{26}\right)\right]\left|\frac{du^+}{dy^+}\right| \qquad [76]$$

has been adopted by Henstock & Hanratty (1976). In [76] ϵ is the eddy viscosity.

In full pipe flow this equation holds for the entire cross-section. In a partially filled pipe, the presence of a free surface may induce appreciable changes in the structure of turbulent eddies. However, considering that correlation methods based on the use of eddy viscosity equations derived for full pipe flow successfully predict the mean film thickness in channel flow, [76] will also be adopted for the present computations.

The Navier–Stokes equation relative to turbulent flow of a liquid film in a channel or in a vertical pipe can easily be integrated by a numerical method, when [76] is used for the eddy viscosity. The liquid Re can then be computed from the velocity profile, and the dimensionless thickness, defined by [40], can be expressed as a function of the Re.

Henstock & Hanratty (1976) followed this approach for annular flow in a vertical pipe. Their results, relative to free-falling flow down an inclined plane at very high Re, can be expressed as

$$m_T^+ = 0.0379\,\mathrm{Re}^{0.9}, \qquad [77]$$

where m_T^+ is the film thickness in turbulent flow.

Equation [77] is directly obtained from [56], using for the friction factor the approximate correlation valid for large Re:

$$f = 0.046\,\mathrm{Re}^{-0.2}. \qquad [78]$$

Equation [77] can be extended to the case of turbulent flow of a liquid in a partially filled pipe. The numerical solution of the Navier–Stokes equation, relative to turbulent flow in a partially filled pipe, is simple in the two cases already considered for laminar flow ($h/D = 0.5$ and $h/D \to 0$), as in these cases the partial differential equation can be transformed into an ordinary differential equation.

In figure 4 the solution obtained for $h/D = 0.5$ is compared with the computations relative to channel flow. As can be seen from this figure, at large Re the solutions relative to pipe and channel flow tend to coincide. At low Re, turbulent solutions approach laminar flow equations [44] and [51].

The stability analysis will be carried out, for channel and pipe flow, only in the limiting case of large Re, making use of [77]. In both cases the shape factor Γ_P is assumed to be equal to one.

Equation [77], for channel flow, can be rewritten as

$$\tau_W = 0.00575\,\frac{\mu U_A}{h}\,\mathrm{Re}^{0.8}, \qquad [79]$$

hence, making use of [9]:

$$\frac{\hat{\tau}_W}{\bar{\tau}_W} = \left(1.8\,\frac{C_R}{\bar{U}_A} - 2\right)\frac{\hat{h}}{\bar{h}}. \qquad [80]$$

This equation substituted into [11] gives the wave velocity

$$\frac{C_R}{\bar{U}_A} = \frac{5}{3}. \qquad [81]$$

The Fr at neutral stability is then obtained from [10]:

$$\mathrm{Fr} = \frac{3}{2}\sqrt{\sin\Theta + \frac{\sigma\alpha^2}{g\rho}}. \qquad [82]$$

A similar procedure is used for pipe flow. Expressing [77] as

$$\tau_W = 0.00575\,\frac{\mu S U_P}{A}\,\mathrm{Re}^{0.8}, \qquad [83]$$

the following relation is derived from [27]:

$$\frac{\hat{\tau}_W}{\bar{\tau}_W} = \left(1.8\,\frac{C_R}{\bar{U}_P} - 2 + 0.2\,\frac{\hat{S}\bar{A}}{\hat{A}\bar{S}}\right)\frac{\hat{A}}{\bar{A}}. \qquad [84]$$

The wave velocity is obtained substituting the latter relation into [29]:

$$\frac{C_R}{\bar{U}_P} = \frac{5}{3} - \frac{2}{3}\,\frac{\bar{A}\hat{S}}{\hat{A}\bar{S}}. \qquad [85]$$

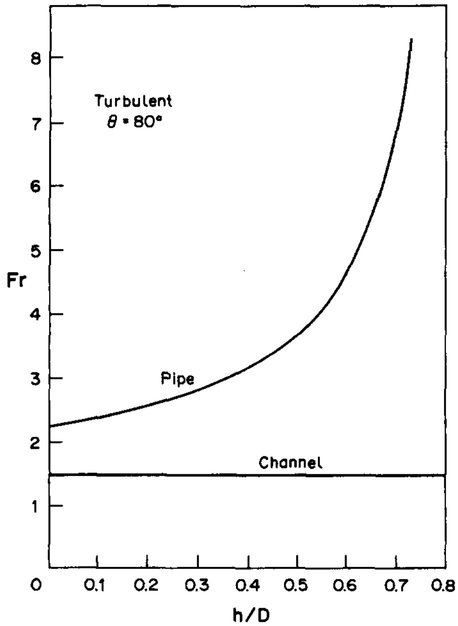

Figure 5. Flow stability in pipe and channel flow; turbulent case.

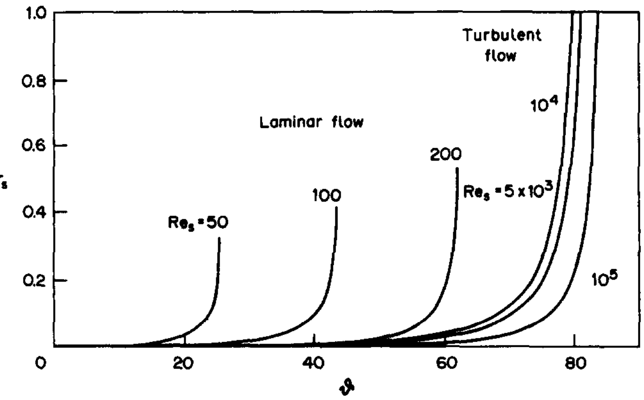

Figure 6. Superficial Fr at neutral stability for laminar and turbulent flow.

Finally, making use of [28], the condition of neutral stability can be expressed as

$$\mathrm{Fr} = \sqrt{\frac{\dfrac{\bar{S}}{\bar{S}_{\mathrm{I}}}\left(\sin\Theta + \dfrac{\sigma\alpha^2}{\rho g}\right)}{\left[\left(\dfrac{C_{\mathrm{R}}}{\bar{U}_{\mathrm{P}}}\right)^2 + 1 - 2\left(\dfrac{C_{\mathrm{R}}}{\bar{U}_{\mathrm{P}}}\right)\right]}}. \qquad [86]$$

Stability conditions in channel and pipe turbulent flow are compared in figure 5. Also in this case, figure 5 shows that there is a limiting value of the film height for which the flow is always stable.

4. CONCLUSIONS

The 2-D problem of steady liquid flow down an inclined tube has been solved by approximate equations valid for laminar (see [55]) and turbulent flow (see [77]). The approximate solution obtained for laminar flow is a very good approximation of the exact solution for the full range of film heights.

The stability analysis has then been based on the integral method proposed by Hanratty & Hershman (1961).

The results of the analysis indicate that the effects of geometry on stability are relevant. In particular, it has been found that:

1. In the limit $h/D \to 0$ the Fr at the transition in pipe flow is more than 3 times larger than in channel flow. In this limit, the stability analysis is exact.
2. For laminar flow down a tube, the liquid layer is always stable for $h/D = 0.5$ ($\hat{\Gamma}_{\mathrm{PR}} = 0$, $\sigma = 0$).
3. A similar result is obtained for fully turbulent flow at $h/D = 0.812$.

These results are summarized in terms of variables known *a priori* in figure 6. In this figure, the superficial $\mathrm{Fr_S}$ at neutral stability, defined as

$$\mathrm{Fr_S} = \frac{U_{\mathrm{S}}}{\sqrt{gD}}, \qquad [87]$$

is represented, both for laminar and turbulent flow, as a function of pipe inclination, Θ, with the superficial Reynolds number, $\mathrm{Re_S}$, defined as

$$\mathrm{Re_S} = \frac{U_S D}{\nu}, \qquad [88]$$

as a parameter.

In [87] and [88] U_S is the superficial velocity (volumetric flow rate per unit cross-section).

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